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Squeezing in a detuned parametric amplifier

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Abstract. The effect of detuning on the squeezing obtained in a prototype model for squeezing, namely the degenerate parametric amplifier, is investigated. A squeezed minimum uncertainty state is only obtained in a set of variables related to the original variables by a time dependent transformation. The required transformation is obtained using a symplectic decomposition of the dynamic evolution matrix.

1. Introduction

Recently there has been much discussion of a special class of oscillator minimum uncertainty states known as squeezed states (Robinson 1975, Stoler 1970, 1971, Lu 1971, 1972, Yuen 1976 and Hollenhorst 1979). These are states which have reduced fluctuations in one field quadrature, when compared with coherent states. Squeezed states are also known as two-photon coherent states.

A number of possible exciting applications of squeezed states have recently been suggested. One application proposes using squeezed states of light in optical communication systems to give a signal-to-noise ratio better than the quantum limit for coherent light (Yuen and Shapiro 1978, Shapiro *et al* 1979). Another suggested application is in the laser interferometric detection of gravitational radiation (Caves 1981).

In view of such applications it is obviously of great importance to devise feasible experimentally realisable schemes to generate squeezed states. Among the earliest models for squeezed-state generation was the degenerate parametric amplifier (Stoler 1970, 1971). Previous analysis of squeezing in this model has been restricted to the case of resonance between the pump and the degenerate signal/idler modes (Milburn and Walls 1981, Lugiato and Strini 1982, Milburn and Walls 1983). It is the object of this paper to consider the consequences of a non-resonant interaction.

2. Properties of squeezed states

We shall briefly describe the mathematical properties of squeezed states. A squeezed state may be defined as follows (Caves 1981),

$$|\alpha, \xi\rangle = D(\alpha)S(\xi)|0\rangle \tag{1}$$

where $S(\xi)$ is the squeeze operator defined by

$$S(\xi) = \exp(\tfrac{1}{2}\xi^* a^2 - \tfrac{1}{2}\xi a^{\dagger 2}) \tag{2}$$

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where $\xi = re^{i\theta}$ and $D(\alpha)$ is the displacement operator

$$D(\alpha) = \exp(\alpha a^\dagger - \alpha^* a) \tag{3}$$

and a, $a^\dagger$ are bose annihilation and creation operators, $[a, a^\dagger] = 1$.

If we now define the quadrature phase operators $\hat{X}_1$, $\hat{X}_2$ by

$$a = \hat{X}_1 + i\hat{X}_2 \tag{4}$$

then $\hat{X}_i$ obey the usual canonical commutation relation $[\hat{X}_1, \hat{X}_2] = i/2$ and $\hat{X}_i = \hat{X}_i^\dagger$. If we further define the variances $V(\hat{X}_i)$ by

$$V(\hat{X}_i) \equiv \langle X_i^2 \rangle - \langle \hat{X}_i \rangle^2 \tag{5}$$

we see that the quadrature phase operators obey the uncertainty relation

$$V(\hat{X}_1) V(\hat{X}_2) \geqslant 1/16 \tag{6}$$

while the coherent states have $V(\hat{X}_1) = V(\hat{X}_2) = 1/4$, the squeezed states $|\alpha, \xi\rangle$ have

$$V(\hat{Y}_1) = \tfrac{1}{4} e^{-2r} \qquad V(\hat{Y}_2) = \tfrac{1}{4} e^{2r} \tag{7}$$

where $\hat{Y}_1 + i\hat{Y}_2 = (\hat{X}_1 + i\hat{X}_2) e^{-i\theta/2}$ is a bose operator corresponding to a rotated complex amplitude (see appendix) and $r = |\xi|$. We note that the squeezing is not obtained directly in the original variables $\hat{X}_i$. This is a characteristic feature of general squeezed states.

A more general mathematical description of squeezed states in n dimensions considers squeezed states to be a subset of the generalised coherent states for the symplectic group $\mathrm{Sp}(2n; \mathbb{R})$ (Milburn 1984).

A simple pictorial representation of a squeezed state may be obtained in terms of the corresponding Wigner function.

In the appendix we show that the Wigner function for a squeezed state has the form

$$W(x_1, x_2) = (2/\pi) \exp\{-2[(x_1 - \bar{x}_1)^2 e^{+2r} + (x_2 - \bar{x}_2)^2 e^{-2r}]\} = (2/\pi) \exp\{-Q/2\} \tag{8}$$

where $\bar{x}_i = \langle \hat{X}_i \rangle$.

If we plot the contour of $W(x_1, x_2)$ defined by $Q = 1$, we obtain an ellipse in (x_1, x_2) space centred on $(\bar{x}_1, \bar{x}_2)$ and with minor axis equal to $V(\hat{X}_1)^{1/2}$ and major axis equal to $V(\hat{X}_2)^{1/2}$. Such a diagram is referred to as a complex amplitude diagram for the squeezed state. In figure 1(c) we have drawn the complex amplitude diagram for the state $|0, \xi\rangle$.

3. Parametric amplifier with detuning

The prototypical system expected to produce squeezed states is the degenerate parametric oscillator. This system may be modelled by the Hamiltonian

$$H = \hbar\omega_0 a^\dagger a + \hbar\chi(a^2 \varepsilon\, e^{i\omega t} + a^{\dagger 2} \varepsilon^* e^{-i\omega t}) \tag{9}$$

where ω_0 is the frequency of the degenerate signal/idler mode, ε is the classical pump field of frequency ω, χ is the coupling constant between the pump and signal modes and a is the annihilation operator for the signal mode. If we choose the phase of the

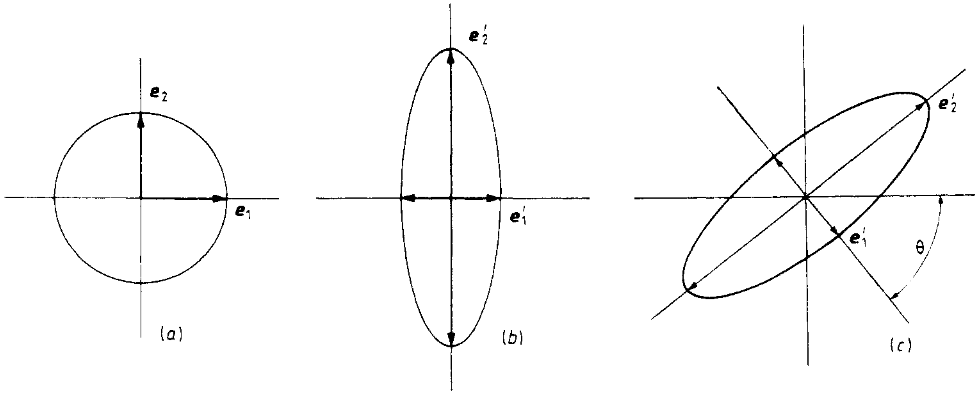

Figure 1. Complex amplitude diagram for a vacuum state (a), a squeezed zero-amplitude state (b), and a rotated squeezed state (c).

driving field so that $\chi\varepsilon$ is purely imaginary ($\chi\varepsilon = -\mathrm{i}|\chi\varepsilon|$) the interaction may be written

$$H_\mathrm{I} = -\mathrm{i}\hbar\kappa(a^2\,\mathrm{e}^{\mathrm{i}\omega t} - a^{\dagger 2}\,\mathrm{e}^{-\mathrm{i}\omega t}) \tag{10}$$

where $\kappa = |\chi\varepsilon|$.

In the degenerate case $\omega = \omega_0$, the Hamiltonian (9) generates ideal squeezed states in the a, $a^\dagger$ variables with squeeze parameter $r = 2\kappa t$. It is the purpose of this paper to investigate the behaviour of squeezing away from resonance.

If we define the variables $\bar{a}$, $\bar{a}^\dagger$ by

$$a = \exp(-\mathrm{i}\omega t/2)\bar{a} \tag{11}$$

the equations of motion may be written

$$\mathrm{d}\bar{a}/\mathrm{d}t = \mathrm{i}/2\Delta\omega\bar{a} + 2\kappa\bar{a}^\dagger \qquad \mathrm{d}\bar{a}^\dagger/\mathrm{d}t = -\mathrm{i}/2\Delta\omega\bar{a}^\dagger + 2\kappa\bar{a} \tag{12}$$

where

$$\Delta\omega = \omega - 2\omega_0. \tag{13}$$

Defining the new quadrature phase operators by

$$\hat{X}_1 = \tfrac{1}{2}(\bar{a} + \bar{a}^\dagger) \qquad \hat{X}_2 = (1/2\mathrm{i})(\bar{a} - \bar{a}^\dagger) \tag{14}$$

and the row vector $\hat{\boldsymbol{X}}^\mathrm{T}(t) = (\hat{X}_1(t), \hat{X}_2(t))$ we have

$$(\mathrm{d}/\mathrm{d}t)\hat{\boldsymbol{X}}(t) = A\hat{\boldsymbol{X}}(t) \tag{15}$$

where

$$A = \begin{pmatrix} 2\kappa & -\Delta\omega/2 \\ \Delta\omega/2 & -2\kappa \end{pmatrix}. \tag{16}$$

Clearly if $\Delta\omega = 0$ the two quadratures evolve independently with $\hat{X}_1$ growing exponentially and $\hat{X}_2$ decaying exponentially. The $\hat{X}_2$ quadrature will be squeezed while the fluctuations in $\hat{X}_1$ will grow. With $\Delta\omega \neq 0$ these two quadratures become coupled and we might expect a feedback of fluctuations of $\hat{X}_1$ into $\hat{X}_2$ destroying or limiting the squeezing.

The solutions to equations (13) may be written in the form

$$\hat{\boldsymbol{X}}(\tau) = T(\gamma, \tau)\hat{\boldsymbol{X}}(0) \tag{17}$$

where

$$T = \begin{pmatrix} \cosh \gamma\tau + \dfrac{1}{\gamma}\sinh \gamma\tau & -\dfrac{(1-\gamma^2)^{1/2}}{\gamma}\sinh \gamma\tau \\ \dfrac{(1-\gamma^2)^{1/2}}{\gamma}\sinh \gamma\tau & \cosh \gamma\tau - \dfrac{1}{\gamma}\sinh \gamma\tau \end{pmatrix} \tag{18}$$

and

$$\tau = \kappa t \qquad \gamma = (1-\delta^2)^{1/2} \qquad \delta = \Delta\omega/4\kappa$$

and $(1-\gamma^2)^{1/2}$ has the sign of δ.

The linear transformation in equation (18) is easily seen to be symplectic (i.e. it preserves the commutation relations). The most general symplectic transformation in one dimension may be written as a rotation followed by a scale change and another rotation (Moshinsky 1973)

$$R(\theta)L(r)R(\psi) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} e^r & 0 \\ 0 & e^{-r} \end{pmatrix}\begin{pmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{pmatrix}. \tag{19}$$

We find here that

$$T(\tau, \gamma) = R(\theta)L(r)R(\theta) \tag{20}$$

where

$$\sinh r = \sinh \gamma\tau/\gamma \tag{21}$$

$$\tan 2\theta = \frac{(1-\gamma^2)^{1/2}}{\gamma}\tanh \gamma\tau. \tag{22}$$

Equivalently equation (17) may be written as

$$\hat{\boldsymbol{X}}(\tau) = U^\dagger(\tau)\hat{\boldsymbol{X}}(0)U(\tau) \tag{23}$$

where

$$\begin{aligned} U(\tau) &= \exp[(-\mathrm{i}\theta/2)(aa^\dagger + a^\dagger a)]\exp[(-r/2)(a^2 - a^{\dagger 2})]\exp[(-\mathrm{i}\theta/2)(aa^\dagger + a^\dagger a)] \\ &\equiv U_R(\theta)S(-r)U_R(\theta). \end{aligned} \tag{24}$$

The time development operator $U(\tau)$ may be written in the form

$$U(\tau) = S(\xi)\exp[-\mathrm{i}\theta(aa^\dagger + a^\dagger a)] \tag{25}$$

where $S(\xi)$ is the general squeeze operator defined in (2) with $\xi = -r\,\mathrm{e}^{-2\mathrm{i}\theta}$. The second factor corresponds to rotations of the canonical variables (see appendix).

If the system begins in the ground state $|0\rangle$, the state of the system at time τ, $|\psi(\tau)\rangle$, is given by

$$|\psi(\tau)\rangle = U(\tau)|0\rangle = \mathrm{e}^{-\mathrm{i}\theta}|0, \xi\rangle. \tag{26}$$

In (26) we have used the fact that the rotation operator acting on $|0\rangle$ produces only a change in phase. The state of the system is thus a rotated squeezed state (see appendix).

The conditions for any unitary transformation to produce a minimum uncertainty state from the vacuum are given in the appendix. Direct comparison of equation (18) with these conditions shows that $|\psi(\tau)\rangle$ will not be a minimum uncertainty state in the original variables $\hat{\boldsymbol{X}}$. This is also clear from the complex amplitude diagram for a rotated squeezed state (figure 1(*c*)). The projection of the ellipse axes onto the original basis must yield uncertainties greater than is possible for a minimum uncertainty state. This implies that the variables for which $|\psi(\tau)\rangle$ is a minimum uncertainty state are rotated by an angle $-\theta$ with respect to the original variables $\hat{X}_1, \hat{X}_2$. We now investigate the squeezing in $|\psi(\tau)\rangle$ in detail.

4. Time dependence of squeezing in a detuned parametric amplifier

(i) $\delta^2 \leqslant 1$

We see from equations (21) and (22) that both the squeeze parameter and the rotation angle of the error ellipse with respect to the original frame are functions of time. In the long-time limit, $\tau \to \infty$, we find that the angle of rotation $\theta(t)$ approaches the limiting value θ_∞ given by

$$\tan 2\theta_\infty = (1-\gamma^2)^{1/2}/\gamma = \delta/(1-\delta^2)^{1/2} \tag{27}$$

while the squeeze parameter approaches r_∞,

$$r_\infty \simeq \gamma\tau - \ln \gamma.$$

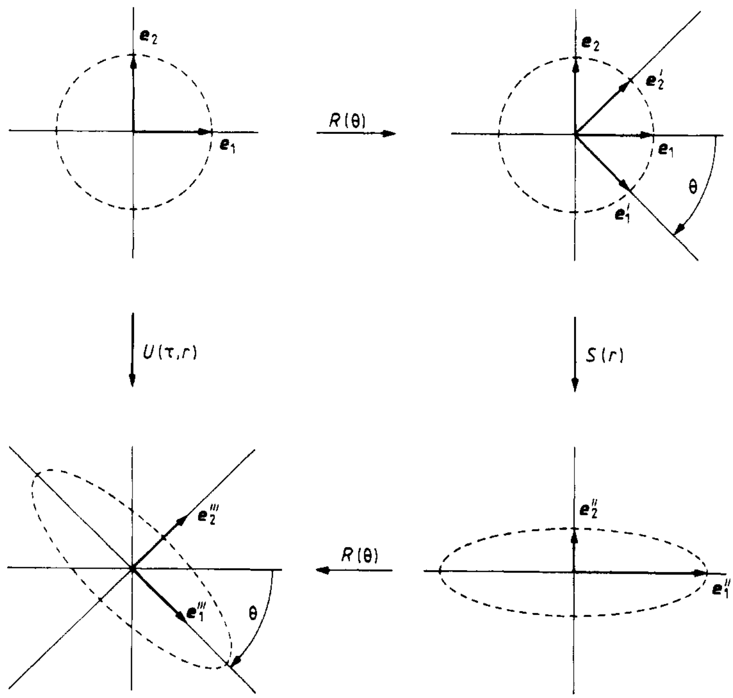

Figure 2. Action of $U(\tau)$ on the vacuum state $|0\rangle$.

Thus eventually the reduction of fluctuations is found in a frame rotated by a constant angle $-\theta_\infty$ with respect to the original variables $(\hat{\boldsymbol{X}}_1, \hat{\boldsymbol{X}}_2)$ (see figure 3).

In the original variables $\hat{\boldsymbol{X}}_1, \hat{\boldsymbol{X}}_2$ the reduction of fluctuations occurs in the $\hat{\boldsymbol{X}}_2$ quadrature over a limited interval of time. Explicitly we have

$$V(\hat{\boldsymbol{X}}_2(\tau)) = \tfrac{1}{4}\{\cosh^2\gamma\tau + [(2-\gamma^2)/\gamma^2]\sinh^2\gamma\tau - (2/\gamma)\cosh\gamma\tau\sinh\gamma\tau\}.$$

This function decreases from 1/4 at $\tau=0$ to a minimum of

$$V(\hat{X}_2(\tau_m)) = \frac{1}{4}\frac{|\delta|}{1+|\delta|} \tag{28}$$

at time

$$\tau_m = \frac{1}{4\gamma}\ln\left\{\frac{1+\gamma}{1-\gamma}\right\}. \tag{29}$$

It then increases monotonically for $\tau > \tau_m$ and is again equal to 1/4 at $\tau = 2\tau_m$. In figure 4 we have plotted $V(\hat{X}_2(\tau))$ against τ for $\gamma = 0.9$ and $\kappa = 1.0$, which demonstrates this behaviour.

The variances of the rotated variables $\hat{X}'_1, \hat{X}'_2$ are given explicitly by

$$V(\hat{X}'_1(\tau)) = \tfrac{1}{4}\gamma^{-2}[(\gamma^2+\sinh^2\gamma\tau)^{1/2} + \sinh\gamma\tau]^2 = \tfrac{1}{4}e^{2r} \tag{30}$$

$$V(\hat{X}'_2(\tau)) = \tfrac{1}{4}\gamma^{-2}[(\gamma^2+\sinh^2\gamma\tau)^{1/2} - \sinh\gamma\tau]^2 = \tfrac{1}{4}e^{-2r} \tag{31}$$

where r is given by (21).

Clearly $V(\hat{X}'_1(\tau))V(\hat{X}'_2(\tau)) = 1/16$. As expected we have a minimum uncertainty state in the rotated variables.

(ii) $\delta^2 > 1$

The solutions for $\delta^2 > 1$ are easily obtained by the replacements $\gamma = \mathrm{i}\lambda$ and $\lambda = (\delta^2-1)^{1/2}$ as $\hat{\boldsymbol{X}}(\tau) = Q(\tau,\lambda)\hat{\boldsymbol{X}}(0)$ with

$$Q(\tau,\lambda) = \begin{pmatrix} \cos\lambda\tau + \dfrac{\sin\lambda\tau}{\lambda} & \dfrac{-(1+\lambda^2)^{1/2}}{\lambda}\sin\lambda\tau \\ \dfrac{(1+\lambda^2)^{1/2}}{\lambda}\sin\lambda\tau & \cos\lambda\tau - \dfrac{\sin\lambda\tau}{\lambda} \end{pmatrix} \tag{32}$$

where $(1+\lambda^2)^{1/2}$ has the sign of δ.

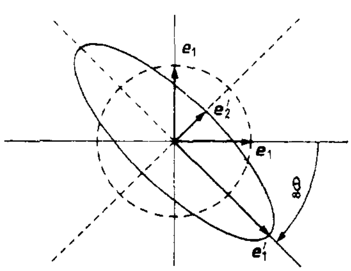

Figure 3. Error ellipse in the long-time limit $\delta^2 \leq 1$.

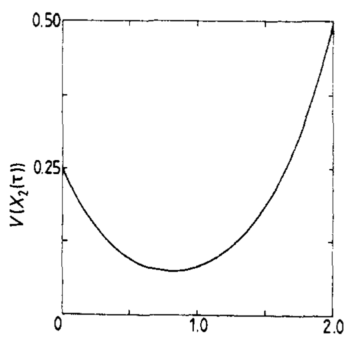

Figure 4. $V(\hat{X}_2(\tau))$ against τ; $\gamma = 0.9$, $\kappa = 1.0$.

The symplectic decomposition is once again $Q(\tau, \lambda) = R(\theta)L(\gamma)R(\theta)$, with

$$\sinh \gamma = \sin \lambda\tau/\lambda \tag{33}$$

and

$$\tan 2\theta = \frac{(\lambda^2+1)^{1/2}}{\lambda} \tan \lambda\tau. \tag{34}$$

We now find that both the squeeze parameter and rotation angle are oscillating functions of time. At $t = n\pi/\lambda$ the squeezing is zero and the system is in a coherent state.

The frame which carries reduced fluctuations is rotated by an angle $-\theta$ with respect to the original frame, where

$$\sin 2\theta = \frac{(1+\lambda^2)^{1/2} \sin \lambda\tau}{[(1+\lambda^2)\sin^2 \lambda\tau + \lambda^2 \cos^2 \lambda\tau]^{1/2}} \tag{35}$$

$$\cos 2\theta = \frac{\lambda \cos \lambda\tau}{[(1+\lambda^2)\sin^2 \lambda\tau + \lambda^2 \cos^2 \lambda\tau]^{1/2}}. \tag{36}$$

We now consider the evolution over one period. At $\tau = 0$ the system is in a coherent state. When τ goes from $0 \to \pi/2\lambda$, the error ellipse rotates in a clockwise direction (see figure 5(*b*)) and has maximum squeezing in $\hat{X}_2'$ at $\tau = \pi/2\lambda$ at an angle $\theta = -\pi/4$ with respect to the original frame. As τ goes from $\pi/2\lambda \to \pi/\lambda$ the ellipse continues to rotate but 'unsqueezes' so that at $\tau = \pi/\lambda$ the system has returned to a coherent state (figure 5(*c*)). From $\tau = \pi/\lambda$ to $\tau = 3\pi/2\lambda$, the system squeezes again. The reduction of fluctuations now appears in $\hat{X}_1'$ but, due to the continued rotation, the error ellipse is in the same orientation with respect to the original frame at $\tau = 3\pi/2\lambda$ as it was at $\tau = \pi/2\lambda$ (figure 5(*d*)). Finally the system returns once again to a coherent state (figure 5(*e*)). The optimum squeezing is at $\theta = \pi/4$.

In the original variables the variances are given by

$$\begin{aligned} V(\hat{X}_1(\tau)) &= (1/4\lambda^2)[(\lambda^2+1) - (\cos 2\lambda\tau + \lambda \sin 2\lambda\tau)] \\ V(\hat{X}_2(\tau)) &= (1/4\lambda^2)[(\lambda^2+1) - (\cos 2\lambda\tau - \lambda \sin 2\lambda\tau)]. \end{aligned} \tag{37}$$

The maximum reduction in fluctuations in these variables is given by

$$V(\hat{X}_1(\tau_m))_{\min} = \frac{1}{4}\frac{|\delta|}{|\delta|+1} \tag{38}$$

where τ_m is such that

$$\cos 2\tau_m = 1/|(\lambda^2+1)^{1/2}| \qquad \sin 2\tau_m = -\lambda/|(\lambda^2+1)^{1/2}|$$

while for the $\hat{X}_2$ quadrature

$$V(\hat{X}_2(\tau_m))_{\min} = \frac{1}{4}\frac{|\delta|}{|\delta|+1} \tag{39}$$

where

$$\cos 2\tau_m = 1/|(\lambda^2+1)^{1/2}| \qquad \sin 2\tau_m = \lambda/|(\lambda^2+1)^{1/2}|.$$

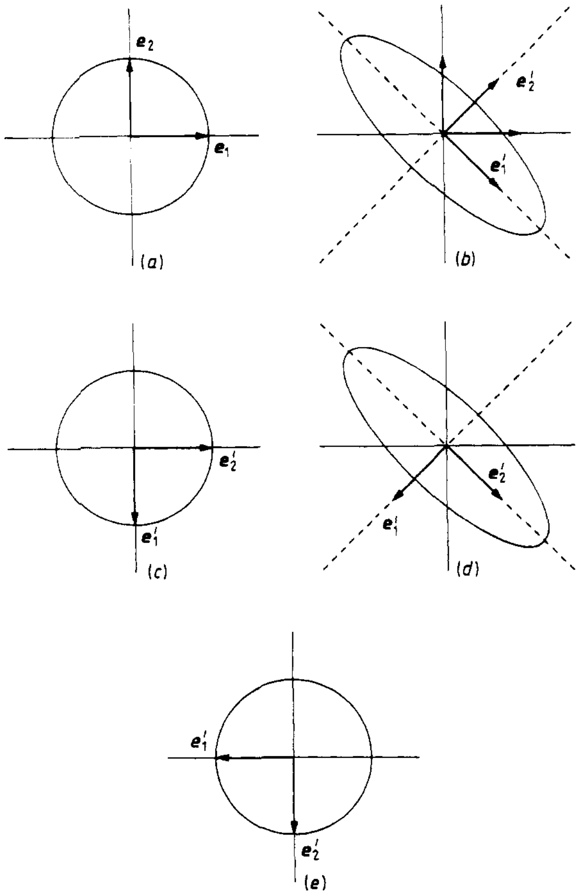

Figure 5. Schematic time dependence of error volume for $\delta > 1$: (*a*) $\tau = 0$; (*b*) $\tau = \pi/2\lambda$; (*c*) $\tau = \pi/\lambda$ and (*d*) $\tau = 3\pi/2\lambda$.

In the $\hat{\boldsymbol{X}}'$ variables the greatest reduction of fluctuations occurs in $\hat{X}_2'$ at time $\tau_m = \pi/2\lambda + n2\pi/\lambda$ and is given by

$$V(\hat{X}_2'(\tau_m))_{\min} = \frac{1}{4}\frac{|\delta|-1}{|\delta|+1} \tag{40}$$

while at time $\tau = 3\pi/2\lambda + n2\pi/\lambda$, $\hat{X}_1'$ carries the greatest reduction in fluctuations with the same minimum value as that given for $V(\hat{X}_2')$ in equation (40).

To summarise, let us consider the maximum reduction in fluctuations occurring in the original variables. We shall take $\hat{X}_2$. For all δ we have that $V(\hat{X}_2)$ has a minimum of $|\delta|/4(1+|\delta|)$. When $\delta = 0$ the minimum value is zero as expected. As the pump field is detuned further from resonance the squeezing diminishes, eventually approaching the coherent state value of $1/4$. Detuning then, always diminishes the reduction of fluctuations obtainable in the original variables. However, as we have shown, there always exists a transformed set of variables for which the reduction of fluctuations is greater than that in the original variables. Comparing equations (44) and (45) we see that for $\delta > 1$ the transformed variable $\hat{X}_2'$ has a minimum variance $0.25/(1+|\delta|)$

below that obtainable in the original variable $\hat{X}_2$. For $\delta^2 < 1$ the fluctuations in the transformed variable $\hat{X}'_2$ can be made arbitrarily small for large τ. It should be noted though that the necessary rotation to $\hat{X}'_2$ is itself time dependent.

Acknowledgment

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Appendix

In this appendix we wish to state, firstly, under what conditions linear canonical transformations produce minimum uncertainty states and secondly how such transformations induce transformations in the complex amplitude diagram.

Let U be a unitary representation of an element of $\mathrm{Sp}(2n\colon \mathbb{R})$, the group of linear canonical transformations. We then define the matrices M by

$$U^{\dagger}\hat{\boldsymbol{X}}U = M\hat{\boldsymbol{X}} \tag{A1}$$

where

$$\hat{\boldsymbol{X}}^{\mathrm{T}} = (\hat{X}_1^1, \hat{X}_1^2 \ldots \hat{X}_1^n, \hat{X}_2^1, \hat{X}_2^2, \ldots \hat{X}_2^n)$$

$$M = \begin{pmatrix} A & C \\ B & D \end{pmatrix} \tag{A2}$$

and A, B, C, D are real $n \times n$ matrices. The state $|\psi\rangle = U|0\rangle$ is a minimum uncertainty state in $\hat{\boldsymbol{X}}$ if and only if a real diagonal matrix Λ exists such that (Milburn 1983)

$$A = D\Lambda \qquad C = -B\Lambda. \tag{A3), (A4}$$

We may obtain a simple pictorial understanding of why some transformations do not produce minimum uncertainty states for a given $\hat{\boldsymbol{X}}$, by considering how the transformations effect the Wigner function phase space (i.e. the complex amplitude diagram). For simplicity we shall consider the one-dimensional case.

The definition of the Wigner function $W(\boldsymbol{x})$ for a state ρ is well known (Louisell 1973). In one dimension it may be written

$$W(\boldsymbol{x}) = \frac{1}{4\pi^2}\int \exp(-\mathrm{i}\boldsymbol{x}^{\mathrm{T}} \cdot \boldsymbol{u})c^{w}(\boldsymbol{u})\,\mathrm{d}\boldsymbol{u} \tag{A5}$$

where

$$c^{w}(\boldsymbol{u}) = \mathrm{Tr}\{\rho \exp(\mathrm{i}\boldsymbol{u}^{\mathrm{T}} \cdot \hat{\boldsymbol{X}})\} \tag{A6}$$

is the characteristic function, and where

$$\boldsymbol{u}^{\mathrm{T}} = (u_1, u_2) \tag{A7}$$

$$\hat{\boldsymbol{X}}^{\mathrm{T}} = (\hat{X}_1, \hat{X}_2). \tag{A8}$$

The Wigner function for the oscillator ground state ($\rho = |0\rangle\langle 0|$) is easily shown to be

$$W(x_1, x_2) = (2/\pi)\exp[-\tfrac{1}{2}\boldsymbol{x}^{\mathrm{T}}A_0^{-1}\boldsymbol{x}] \tag{A9}$$

where

$$A_0 = \tfrac{1}{4}I. \tag{A10}$$

The complex amplitude diagram for $|0\rangle$ is then obtained by plotting the quadratic form $\boldsymbol{x}^{\mathrm{T}}A_0^{-1}\boldsymbol{x} = 1$ i.e. $x_1^2 + x_2^2 = \frac{1}{4}$. This is simply a circle centred on the origin.

Consider the state $|\psi\rangle = U|0\rangle$ where U is a unitary operator corresponding to a linear canonical transformation. We now find the Wigner function for $|\psi\rangle$ from the Wigner function for $|0\rangle$ by determining how U transforms points in $\{x_1, x_2\}$ space. The characteristic function is given by

$$c^{w}(\boldsymbol{u}) = \langle 0|U^{+}\exp(\mathrm{i}\boldsymbol{u}^{\mathrm{T}}\cdot\hat{\boldsymbol{X}})U|0\rangle.$$

Now define the matrix M by

$$\hat{\boldsymbol{X}}' = U^{+}\hat{\boldsymbol{X}}U \equiv M\hat{\boldsymbol{X}}. \tag{A11}$$

Then

$$c^{w}(\boldsymbol{u}) = \langle 0|\exp(\mathrm{i}\boldsymbol{u}'^{\mathrm{T}}\cdot\hat{\boldsymbol{X}})|0\rangle$$

where

$$\boldsymbol{u}' = M^{\mathrm{T}}\boldsymbol{u}. \tag{A12}$$

We easily find that $c^{w}(\boldsymbol{u}) = \exp[-\frac{1}{2}\boldsymbol{u}'^{\mathrm{T}}A_0\boldsymbol{u}']$ with A_0 given in equation (A10).

The Wigner function is then given as

$$W(\boldsymbol{x}) = \frac{1}{4\pi^2}\int \exp(-\mathrm{i}\boldsymbol{x}^{\mathrm{T}}\cdot\boldsymbol{u})\exp\{-\tfrac{1}{2}\boldsymbol{u}'^{\mathrm{T}}A_0\boldsymbol{u}'\}\,\mathrm{d}\boldsymbol{u}.$$

Under the change of variable $\boldsymbol{u}' = M^{\mathrm{T}}\boldsymbol{u}$ this becomes

$$W(\boldsymbol{x}) = \frac{1}{4\pi^2}\int \exp(-\mathrm{i}\boldsymbol{x}'^{\mathrm{T}}\cdot\boldsymbol{u}')\exp[-\tfrac{1}{2}\boldsymbol{u}'^{\mathrm{T}}A_0\boldsymbol{u}']\,\mathrm{d}\boldsymbol{u}'$$

where

$$\boldsymbol{x}' = M^{-1}\boldsymbol{x} \tag{A13}$$

and we have used the fact that the Jacobian of the transformation in equation (A12) is one. We then find

$$W(\boldsymbol{x}) = (2/\pi)\exp[-\tfrac{1}{2}\boldsymbol{x}'^{\mathrm{T}}A_0^{-1}\boldsymbol{x}']. \tag{A14}$$

The quadratic form, which defines the complex amplitude diagram for $|\psi\rangle$ in the new variables $\boldsymbol{x}'$, is still a circle. In terms of the original variables of course it may look quite different.

The interpretation of the quadratic form $\boldsymbol{x}^{\mathrm{T}}A_0^{-1}\boldsymbol{x}$, which defines the complex amplitude diagram, is facilitated by regarding x_1, x_2 as the coordinates of a point A, with respect to the orthogonal unit basis vectors $\boldsymbol{e}_1$, $\boldsymbol{e}_2$. The transformation in (A13) may then be considered as expressing the coordinates of A with respect to a new basis $\boldsymbol{e}_1'$, $\boldsymbol{e}_2'$ where

$$\begin{pmatrix}\boldsymbol{e}_1'\\ \boldsymbol{e}_2'\end{pmatrix} = M^{\mathrm{T}}\begin{pmatrix}\boldsymbol{e}_1\\ \boldsymbol{e}_2\end{pmatrix}. \tag{A15}$$

As an example of the above considerations, consider the squeezed state $|0, \xi\rangle = S(\xi)|0\rangle$ (see (2)). The matrix representation of $S(\xi)$ is

$$M = \begin{pmatrix} \cosh r - \sinh r \cos 2\theta & -\sinh r \sin 2\theta \\ -\sinh r \sin 2\theta & \cosh r + \sinh r \cos 2\theta \end{pmatrix}. \tag{A16}$$

Direct inspection of M together with (A3) shows that $|0, \xi\rangle$ is not a minimum uncertainty state for $\hat{\boldsymbol{X}}$.

We now construct the complex amplitude diagram for $|0, \xi\rangle$. $S(\xi)$ may be disentangled as

$$S(\xi) = U_R(-\theta/2) S(r) U_R(\theta/2) \tag{A17}$$

where the operators are defined in equation (24). The matrix representations of U_R and S are respectively

$$R(\theta/2) = \begin{pmatrix} \cos\theta/2 & -\sin\theta/2 \\ \sin\theta/2 & \cos\theta/2 \end{pmatrix} \tag{A18}$$

$$L(r) = \begin{pmatrix} e^{-r} & 0 \\ 0 & e^{r} \end{pmatrix}. \tag{A19}$$

$R(\theta/2)$ effects a rotation of the basis given by

$$\begin{pmatrix} \boldsymbol{e}'_1 \\ \boldsymbol{e}'_2 \end{pmatrix} = R^{\mathrm{T}} \begin{pmatrix} \boldsymbol{e}_1 \\ \boldsymbol{e}_2 \end{pmatrix} \tag{A20}$$

while $L(r)$ effects a scale change of the basis given by

$$\begin{pmatrix} \boldsymbol{e}'_1 \\ \boldsymbol{e}'_2 \end{pmatrix} = L^{\mathrm{T}} \begin{pmatrix} \boldsymbol{e}_1 \\ \boldsymbol{e}_2 \end{pmatrix}. \tag{A21}$$

While the unit circle in bases related by $R(\theta/2)$ are identical, the unit circle in the basis $\boldsymbol{e}'_1$, $\boldsymbol{e}'_2$, given by (A21), will appear as an ellipse in the original basis $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ (see figure 1(*b*)).

We are now in a position to construct the complex amplitude diagram for the state $|0, \xi\rangle = U_R(-\theta/2)S(r)U_R(\theta/2)|0\rangle$. The state $U_R(\theta/2)|0\rangle$ differs from the vacuum by a phase factor only, thus the complex amplitude diagram for $|0, \xi\rangle$ is identical to that for the state $U_r(-\theta/2)S(r)|0\rangle$. (This is also seen by noting that the circular quadratic form associated with $|0\rangle$ is invariant under rotations.) The state $S(r)|0\rangle$ is represented by a circular quadratic form in a basis contracted along $\boldsymbol{e}_1$ and dilated along $\boldsymbol{e}_2$ (A21). Thus in the original basis $S(r)|0\rangle$ appears as an ellipse (figure 1(*b*)). The final rotation $U_R(-\theta/2)$ acting on $S(r)|0\rangle$ produces a state represented by a circular quadratic form in the scaled basis $\{\boldsymbol{e}'_1, \boldsymbol{e}'_2\}$, but rotated by $-\theta/2$ with respect to the original basis. The result viewed from the original basis is shown in figure 1(*c*). The complex amplitude diagram for the state $|0, \xi\rangle$ is an ellipse rotated by $-\theta/2$ with respect to the original basis.

It is now clear that $|0, \xi\rangle$ will not be a minimum uncertainty state with respect to the original variables $\boldsymbol{x}$. However, it is a minimum uncertainty state with respect to the rotated variables $\boldsymbol{y}$ where

$$\boldsymbol{y} = R(-\theta/2)\boldsymbol{x}$$

or in terms of the corresponding operators

$$\hat{\boldsymbol{Y}} = R(\theta/2)\hat{\boldsymbol{X}}$$

that is

$$\hat{Y}_1 + \mathrm{i}\,\hat{Y}_2 = (\hat{X}_1 + \mathrm{i}\hat{X}_2)\exp(-\mathrm{i}\theta/2). \tag{A22}$$

The explicit form of the Wigner function for the (unrotated) squeezed state $|0, r\rangle$ is (from (A19) and (A14))

$$W = \frac{2}{\pi}\exp\{-2[(x_1 - \bar{x}_1)^2\,\mathrm{e}^{2r} + (x_2 - \bar{x}_2)^2\,\mathrm{e}^{-2r}]\} \tag{A23}$$

where $\bar{x}_i = \langle \hat{x}_i \rangle$.

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